Scheduling of Wind Farms for Optimal Frequency Response and Energy Recovery

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***Abstract*—This paper deals with control of variable speed wind turbines which provide frequency support through temporary overproduction. In particular, it determines the optimal profile of power extraction among multiple generators in order to minimize the total loss of efficiency while allowing for a prescribed increase in generation. Starting with the simplifying assumption of unconstrained generated/supplied power for the single turbine, the scheduling is characterized as the solution of an optimal control problem. On the basis of this result, an heuristic control strategy is proposed for the case of turbines with limited power output, investigating under which conditions this choice achieves optimality. Using a similar approach, the problem of energy recovery is also considered, calculating the optimal power profiles which bring back the turbines to their working point of maximum efficiency after having provided frequency response.**

***Index Terms*—Wind turbines, frequency control, optimal scheduling, monotone systems.**

NOMENCLATURE

C	Power coefficient with fixed pitch angle.
E	Rotational kinetic energy (J).
E^0	Vector of initial kinetic energies (J).
E_{MAX}	Maximum kinetic energy of single turbine (J).
E_{MIN}	Minimum kinetic energy of single turbine (J).
E_{ss}	Kinetic energy of maximum efficiency (J).
J	Moment of inertia ($Kg \cdot m^2$).
N	Number of turbines.
P_L	Min. aggregate power during recovery phase (W).
P_r	Reference for aggregate output power (W).
P_{MAX}	Maximum output power of single turbine (W).
P_{MIN}	Minimum output power of single turbine (W).
R	Turbine radius (m).
T_e	Electrical torque ($N \cdot m$).
T_m	Mechanical torque ($N \cdot m$).
Π	Mechanical power extracted from the wind (W).
Π_E	Energy derivative of mechanical power (s^{-1}).
$\Pi_{E_i}^{-1}$	Inverse of Π_E for the i-th turbine (J).
Π_{TOT}	Total mech. power extracted by the wind farm (W).
$\bar{C}$	Power coefficient.
λ	Tip-speed ratio.
$\mathscr{S}$	Subset of turbines.
μ	Air density (kg/m^3).
ω	Turbine rotor speed (rad/s).
ϕ	Control feedback law with constrained power (W).
θ	Pitch angle of the blades (deg).
φ	Control feedback law with unconstrained power (W).
h	Maximized total mechanical power (W).
v	Wind speed (m/s).
(E^*, P^*)	Scheduling with constrained power (J,W).
$(E^\star, P^\star)$	Optimal scheduling with unconstrained power (J,W).

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I. INTRODUCTION AND MOTIVATIONS

IN the last few years growing environmental concerns and advancements in technology have led to an increasing penetration of wind generation in power systems. In the near future wind turbines will represent a significant component of the total supply and will be required to procure the same services that nowadays are provided by conventional synchronous generators [1]. An element of particular importance and interest is the provision of frequency response: following an outage in the network the wind turbines are required to reduce the resulting fall of frequency by generating an extra quantity of power [2]. Two main approaches have been proposed so far: keeping a power reserve by operating the turbines at a deloaded maximum power curve [3], [4] or releasing part of the kinetic energy stored in the rotating shafts by slowing down the turbines [5]. This is usually achieved by introducing an additional feedback loop in the speed controller of the generator which takes into account the system frequency variation [6]. Other formulations have also been proposed, considering the frequency derivative [7] or introducing additional control loops to improve system damping and emulate synchronizing power characteristics [8].

This paper applies the latter approach of energy overproduction on a higher level: only the mechanical dynamics of the single turbines are considered and the analysis is extended to multiple generators. These are in general affected by different amounts of wind and therefore operate, in steady state, at different rotor speeds. Based on the preliminary results presented in the last part of [9], the frequency support is formulated as an optimal control problem. In case of frequency events, a set-point is introduced for the aggregate extra generation that must be provided by the wind farm, calculating the power profile of each turbine in order to minimize the resulting losses of kinetic energy. In this case the application of the classical tools of optimal control is prevented by the complicated expression that describes the power extraction from the wind and the large number of considered generators. For this reason, the power profile of the turbines is obtained by exploiting particular monotonicity properties that arise if one considers the kinetic energy dynamics and its relationship with the efficiency of the turbines.

The problem is initially solved for the simplified case without constraints on generated power, using the results as a starting point for the more realistic analysis in which the power provided by each turbine cannot exceed some technical limits. It is worth noticing that the optimal power profiles are straightforward to calculate numerically and can be used, in practical implementations, as references for the speed controller of the turbines. Furthermore, a similar approach can be used to study the energy recovery problem: after having provided frequency response, the aggregate power set-point is reduced and turbines are brought back to the operating state of maximum efficiency. With the same tools, it is possible to determine which power references are feasible for a given energy state and calculate the power profiles which allow to perform the recovery in minimal time.

The choice of analyzing frequency support with wind turbines using optimal control techniques is novel and gives significant insights on the structure of the problem. Moreover, it represents a theoretical framework that can be expanded in order to consider, for example, other ancillary services to be provided by the generators. While the improvement obtained with the proposed solution depends in general from the chosen turbine parameters and wind conditions, our technique also provides an upper bound to the maximum achievable efficiency which could be useful to evaluate other control strategies.

The rest of the paper is structured as follows: Section II presents the model of the individual wind generator and the expression for the power extracted from the wind, specifying the assumptions and the context in which the frequency response problem will be formulated. The optimal power profiles are calculated for the unconstrained case in Section III through the resolution of a static optimization problem. The case with constraints on generated power is studied in Section IV and simulation results are presented in Section V. The problem of energy recovery of the turbines and its time minimizing solution are described in Section VI while Section VII contains some final considerations.

II. Modelling of Individual Turbines

Each wind turbine is modelled in its mechanical part as a rotating mass. The dynamics of the rotor speed ω are described by the swing equation:

$$\dot{\omega} = \frac{1}{J}\left(T_m - T_e\right) \tag{1}$$

where J is the total moment of inertia of the rotating shafts, T_m is the mechanical torque resulting from the wind and T_e is the electromagnetic counter torque. The electrical dynamics of the turbine are much faster than the mechanical ones and therefore have been neglected. The additional control loop which determines the electrical quantities of rotor and stator in order to achieve a certain torque T_e is not considered and T_e directly represents the control input of the system.

If we denote by v the wind speed, by R the radius of the rotor and by μ the air density, we obtain the following expressions for the power of the wind P_w and the corresponding mechanical torque T_m acting on the turbine:

$$P_w(v) = \frac{\mu \pi R^2 v^3}{2} \qquad T_m(v,\omega,\theta) = \frac{P_w(v)\bar{C}(\lambda,\theta)}{\omega}\;. \tag{2}$$

The power coefficient $\bar{C}$ represents the fraction of wind power P_w captured by the turbine and depends on the tip-speed ratio $\lambda = \frac{\omega R}{v}$ and the pitch angle of the blades θ. For instance in Section V, where simulations are presented, we will consider the formulation proposed by [10]:

$$\begin{aligned} \bar{C}(\lambda,\theta) &= 0.22\left(\frac{116}{\lambda_i} - 0.4\theta - 5\right)e^{-\frac{12.5}{\lambda_i}} \\ \frac{1}{\lambda_i} &= \frac{1}{\lambda + 0.08\theta} - \frac{0.035}{\theta^3+1}, \end{aligned} \tag{3}$$

however, any nonlinear dependence that fulfills the qualitative assumptions detailed below is suitable for the design techniques discussed in subsequent sections. For the wind speed v and the pitch angle θ, the following assumptions are introduced:

Assumption 1: Given the relatively short time interval to be considered for the frequency response of the turbines, it is reasonable to assume that the wind speed v is constant in time. This is a common assumption for studies in this area [3], [11]. Furthermore, if one excludes high wind conditions, pitch angle actions are not applied and the angle θ is constant and equal to zero. The power coefficient can then be defined exclusively as a function of the rotor speed ω and wind speed v:

$$C(\omega,v) = \bar{C}\left(\frac{\omega R}{v},0\right). \tag{4}$$

A diagram of the wind turbine model, with a representation of the current assumptions, is presented in Fig. 1. The Optimal Power Point Tracking block (OPPT) is the controller which is used in normal operation and determines the reference of electric torque $T_{e_{ref}}$ in order to achieve maximum efficiency. The frequency control strategy presented in the next sections will bypass this block and directly determine T_e when a frequency event occurs.

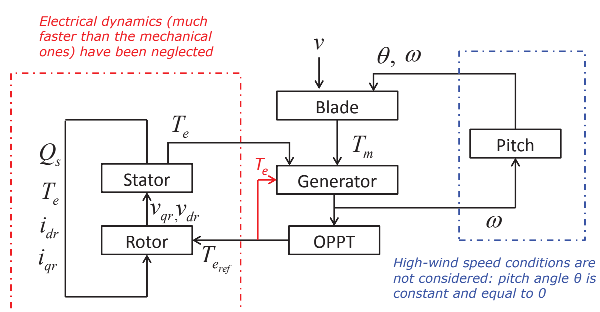

Fig. 1. Block representation of the individual wind turbine. Source: [12].

On the basis of the expressions of $\bar{C}$ proposed in the literature, the following properties are considered for the power coefficient:

Assumption 2: For a fixed wind speed v, it is assumed that the coefficient $C(\omega,v)$ has a unique maximum for $\omega = \omega_{ss}(v)$ and is a monotonic increasing function in some interval $[\omega_L(v), \omega_{ss}(v)]$.

In order to study the optimal control problem to be defined in the next section, it is convenient to introduce a change of coordinates, describing the state evolution of a single turbine by considering its kinetic energy $E = \frac{1}{2}J\omega^2$ (rather than its angular speed). It is straightforward to obtain an expression for the mechanical power P_m that depends only on E and v:

$$P_m = P_w(v)C(\omega, v) = P_w(v)C\left(\sqrt{\frac{2E}{J}}, v\right) = \Pi(E, v). \qquad (5)$$

Values of Π at different wind speeds, for the power coefficient introduced in (3) and the turbine parameters considered in the simulation section, are shown in Fig. 2.

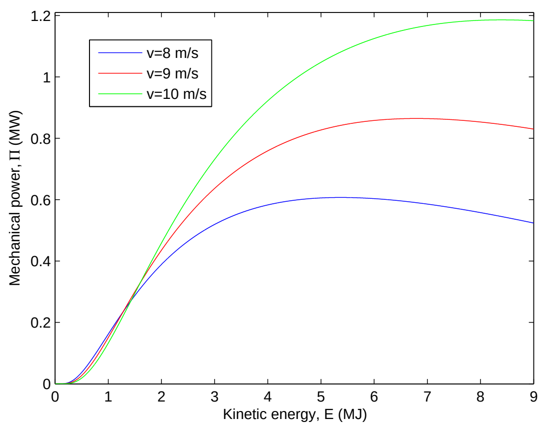

Fig. 2. Mechanical power Π extracted from the wind as a function of kinetic energy E, for different wind speeds v.

The state equation in the new coordinate becomes:

$$\dot{E} = J\omega\dot{\omega} = P_m - P_e = \Pi(E, v) - P_e. \qquad (6)$$

The term $P_e = T_e\omega$ in (6) represents the electrical power generated by the turbine. Assuming that the rotor speed ω can be measured without uncertainties, P_e can be considered as the new input of the system. To take into account the physical limitations of the turbine, constraints are imposed on the kinetic energy E which is limited to some interval $\mathscr{E} = [E_{MIN}, E_{MAX}]$.

Remark 1: The properties of C considered in Assumption 2 have a direct correspondence in the function Π. In particular, for each v, the mechanical power $\Pi(E, v)$ has a unique maximum for $E = E_{ss}(v) = \frac{1}{2}Jw_{ss}^2(v)$ and is monotone increasing in the interval $[E_L(v), E_{ss}(v)]$ with $E_L(v) = \frac{1}{2}Jw_L^2(v)$.

Assumption 3: For any fixed v, given the expressions for the coefficient $\bar{C}$ and the specifications of the turbines found in the literature, it is reasonable to assume that $\Pi(E, v)$ is strictly concave on $[E_L(v), E_{MAX}]$ and the energy E of the single generator, in the overproduction regime of the frequency response, is always within the concavity region of Π:

$$E_L(v) \le E_{MIN} \le E \le E_{ss}(v) \le E_{MAX}. \qquad (7)$$

III. Optimal Frequency Response: The Unconstrained Case

A population of N turbines is considered and the kinetic energy, electrical power and wind speed of the i-th generator are denoted respectively by E_i, P_i and v_i. All turbines are initially operating in steady-state at the kinetic energy $E_{ss}(v_i)$ which guarantees the maximum efficiency, with the following equality holding for the mechanical and electrical power:

$$\Pi(E_{ss}(v_i), v_i) = P_i(0) \qquad i = 1 \dots N. \qquad (8)$$

The wind speeds experienced by the individual turbines are assumed in general to be different: this allows to consider, for example, the wind speed reduction caused in the wind farm by the upstream turbines, also known as *wake effect* [12]. When an outage occurs in the network at time $t = 0$, in order to reduce the resulting frequency drop, the wind farm increases its aggregate generated power by releasing part of the kinetic energy stored in the rotating shafts. In particular, a reference $P_r(\cdot)$ is set on the time interval $[0, T]$, requiring an aggregate power which is greater than the one at steady state:

$$P_r(t) > \textstyle\sum_{i=1}^N \Pi(E_{ss}(v_i), v_i) = P_{ss} \qquad \forall t \in [0, T]. \qquad (9)$$

Our aim is to determine the power profile of each turbine $P_i(\cdot)$ in order to satisfy the following:

$$\begin{aligned} \sum_{i=1}^N P_i(t) = P_r(t) \qquad & \forall t \in [0, T] \\ E_i(t) \in [E_{MIN}, E_{MAX}] \qquad & i = 1 \dots N. \end{aligned} \qquad (10)$$

In general, there exist multiple choices of P_i which are feasible for (10) and it is therefore important to introduce some optimality criterion in the calculation of the power profiles. In this respect, a logical choice is to define as optimal the set of $P_i(\cdot)$ which maximizes the total final energy $\sum_i E_i(T)$ of the turbines. This choice takes into account the following phase of recovery of the generators that, after having provided frequency response, are brought back to their working point of maximum efficiency. Furthermore, it will be shown that the resulting power profiles guarantee feasibility for the largest class of power references P_r.

The simpler case in which no constraints are imposed on the generated power P_i of the turbines is initially analysed. The corresponding optimization problem is:

$$\begin{aligned} \max_{P_i(\cdot), i=1\dots N} \quad & \sum_{i=1}^N E_i(T) \\ \text{s. t.} \quad & \textstyle\sum_{i=1}^N P_i(t) = P_r(t) \\ & E_i(0) = E_i^0 \\ & \dot{E}_i(t) = \Pi(E_i(t), v_i) - P_i(t) \\ & E_i(t) \in [E_{MIN}, E_{MAX}]. \end{aligned} \quad \begin{pmatrix} \forall i = 1, \dots, N \\ \forall t \in [0, T] \end{pmatrix} \qquad (11)$$

In this scenario, given any state vectors E^a and E^b in $\mathscr{E}^N$ of equivalent total energy (viz. such that $\sum_{i=1}^N E_i^a = \sum_{i=1}^N E_i^b$), it is possible to transfer between turbines the amount of energy required so as to achieve an instantaneous switch between the two states. This is true since all P_is are unconstrained and (as we are neglecting power losses deriving from friction in the mechanical components) the total power required for the switch is zero. Therefore, indications on the solution of (11) can be obtained by solving, at each time instant t, a static optimization problem. In particular, the kinetic energy E_{TOT} of the wind farm is distributed among the turbines in order to obtain the maximum value $h(E_{TOT})$ of total mechanical power extracted from the wind:

$$h(E_{TOT}) := \max_{x_i, i=1\dots N} \sum_{i=1}^{N} \Pi(x_i, v_i) \tag{12a}$$

$$\text{s.t.} \sum_{i=1}^{N} x_i = E_{TOT} \tag{12b}$$

$$x_i \in [E_{MIN}, E_{MAX}]. \tag{12c}$$

The general idea is to calculate the optimal solution for (12) (derived in Theorem 1) and then show with Theorem 2 that the optimal trajectories for the original problem (11) can simply be obtained by solving the static optimization problem at each time instant. The corresponding optimal power profiles are straightforward to derive through Proposition 3.

In order to solve (12) it is useful to introduce the partial derivative $\Pi_E(E,v) = \frac{\partial \Pi(E,v)}{\partial E}$ and its inverse function with respect to E when $v = v_i$, denoted by $\Pi_{E_i}^{-1} : [0, \Pi_E(E_{MIN}, v_i)] \to [E_{MIN}, E_{ss}(v_i)]$. It follows from the strict concavity of Π, established in Assumption 3, that $\Pi_{E_i}^{-1}$ is always well defined and monotone decreasing. Since the static problem (12) will be solved in Theorem 1 using Karush-Kuhn-Tucker (KKT) conditions, we preliminary show existence and uniqueness of the quantity $K(E_{TOT})$, which depends on the total kinetic energy E_{TOT} of the turbines and will represent the multiplier associated to the equality constraint.

Proposition 1: For any value of total energy $E_{TOT} \in \left[NE_{MIN}, \sum_{i=1}^{N} E_{ss}(v_i)\right]$ there exists one and only one κ, that we denote by $K(E_{TOT})$, such that the following holds:

$$\sum_{i=1}^{N} \Pi_{E_i}^{-1}\left(\min(\kappa, \Pi_E(E_{MIN}, v_i))\right) = E_{TOT}. \tag{13}$$

Proof: Existence and uniqueness of $K(E_{TOT})$ are straightforward to verify if one considers that the function $K^{-1}(\kappa)$, which denotes the left-hand-side of (13), is monotonic decreasing, continuous and its image includes the interval $[NE_{MIN}, \sum_i E_{ss}(v_i)]$:

$$\begin{aligned} K^{-1}(0) &= \sum_{i=1}^{N} E_{ss}(v_i) \\ K^{-1}\left(\max_{i\in\{1,\dots,N\}} \left(\Pi_E\left(E_{MIN}, v_i\right)\right)\right) &= NE_{MIN}. \end{aligned} \tag{14}$$

■

This result allows to determine the solution of the static optimization problem:

Theorem 1: Under Assumption 3 for the function Π, if the total kinetic energy of the turbines E_{TOT} is such that $E_{TOT} \in [NE_{MIN}, \sum_i E_{ss}(v_i)]$, the solution $x^\star$ for problem (12) exists and is unique. Given $K(E_{TOT})$ introduced in Proposition 1, $x^\star$ has the following expression:

$$x_i^\star = \Pi_{E_i}^{-1}\left(\min(K(E_{TOT}), \Pi_E(E_{MIN}, v_i))\right) \qquad i = 1, \dots, N. \tag{15}$$

Proof: See Appendix A. ■

For a better understanding of the structure of the solution, a graphical representation is provided in Fig. 3 for the simple case of three turbines affected by different wind speeds. Three distinct values of total energy $[\tilde{E}_1, \tilde{E}_2, \tilde{E}_3]$ are considered, associating to each $\tilde{E}_j$ the constant value $K(\tilde{E}_j)$, represented in the figure as a black/grey dashed line. For the i-th generator, if $K(\tilde{E}_j) < \Pi_E(E_{MIN}, v_i)$, the optimal $x_i^\star$ for (12) with $E_{TOT} = \tilde{E}_j$ is such that $\Pi_E(x_i^\star, v_i) = K(\tilde{E}_j)$ (see for example the projections on the x-axis of the red curve). If on the other hand $K(\tilde{E}_j) \geq \Pi_E(E_{MIN}, v_i)$, the corresponding $x_i^\star$ is equal to E_{MIN} (for example the x-value of the curve in green). Notice also that, following the concavity of Π presented in Assumption 3, the derivatives Π_E (shown in Fig. 3 for different wind speeds) are always monotonic decreasing. This means that $\Pi_E(E_{MIN}, v_i)$ will correspond to the maximum rate of change in the mechanical power extracted from the wind.

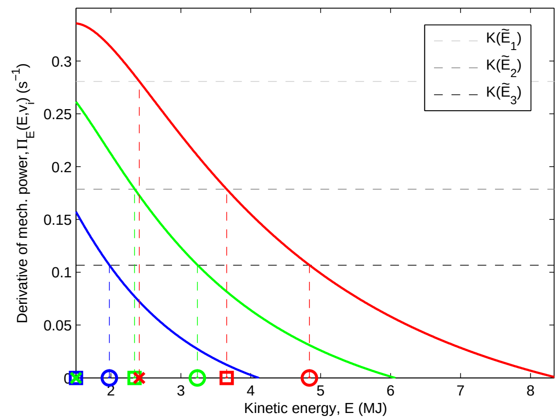

Fig. 3. Solution of the static maximization problem (12) for different values of E_{TOT}. The optimal energy values of each turbine are in the same colour of the corresponding Π_E curves (blue: $v_i = 7m/s$, green: $v_i = 8.5m/s$, red: $v_i = 10m/s$). They are displayed as circles when $E_{TOT} = \tilde{E}_1 = 10MJ$, as squares when $E_{TOT} = \tilde{E}_2 = 7.5MJ$ and as crosses when $E_{TOT} = \tilde{E}_3 = 5.4MJ$.

The following property of the function $h(\cdot)$, as defined in (12), is crucial for determining the optimal solution for the original problem (11):

Proposition 2: The maximum $h(E_{TOT})$ of (12) is strictly concave and Lipschitz continuous with respect to E_{TOT} in the interval $[NE_{MIN}, \sum_{i=1}^{N} E_{ss}(v_i)]$.

Proof: See Appendix B. ■

From the results of Proposition 2, using Picard-Lindelöf Theorem, it is possible to define $E_{TOT}^\star(\cdot)$ as the unique solution of the following ODE:

$$\dot{E}_{TOT}(t) = h(E_{TOT}(t)) - P_r(t) \qquad E_{TOT}(0) = \sum_{i=1}^{N} E_i^0. \tag{16}$$

A constructive solution can then be provided for the problem of final energy maximization:

Theorem 2: The functions $E_i^\star(\cdot)$, $i = [1, \dots, N]$, defined as the solution at each time $t \in [0, T]$ of problem (12) for $E_{TOT} = E_{TOT}^\star(t)$, are optimal state trajectories for the final energy maximization problem (11).

Proof: See Appendix C. ■

An expression for $E^\star$ can be obtained by evaluating the min function in (15) for $E_{TOT} = E_{TOT}^\star$:

$$E_i^\star(t) = \begin{cases} \Pi_{E_i}^{-1}\left(K\left(E_{TOT}^\star(t)\right)\right) & \text{if} \quad i \in \mathscr{S}_1(t) \\ E_{MIN} & \text{if} \quad i \in \mathscr{S}_2(t) \end{cases} \tag{17}$$

where $E^\star_{TOT}(t) = \sum_{i=1}^N E^\star_i(t)$ and the two sets $\mathscr{S}_1(t)$ and $\mathscr{S}_2(t)$ are defined as follows:

$$\begin{aligned}\mathscr{S}_1(t) &= \{i : \Pi_E(E_{MIN}, v_i) \geq K(E^\star_{TOT}(t))\}\\ \mathscr{S}_2(t) &= \{1,2,\ldots,N\} \setminus \mathscr{S}_1(t).\end{aligned} \tag{18}$$

Remark 2: At any time t, the unique optimal solution $E^\star(t)$ is obtained by dividing the turbines in two groups: the ones in $\mathscr{S}_2(t)$ will have minimum energy E_{MIN} while the remaining ones will be characterized by equal derivatives $\Pi_E(E^\star_i(t), v_i) = K(E^\star_{TOT}(t))$. By evaluating Π_E at $E^\star_i$ in (17), considering the definitions of $\mathscr{S}_1$ and $\mathscr{S}_2$, we have:

$$\Pi_E(E^\star_i(t), v_i) \geq \Pi_E(E^\star_j(t), v_j) \ \ \forall i \in \mathscr{S}_1(t) \ \ \forall j \in \mathscr{S}_2(t). \tag{19}$$

We are now interested in determining the power profiles $P^\star$ which generate the optimal state trajectories. The analysis will consider the case in which it is not necessary to perform an instantaneous energy switch since the initial state corresponds to the optimal solution at time $t = 0$:

$$E^0_i = E^\star_i(0) \qquad i = 1,\ldots,N. \tag{20}$$

If this is not the case, it is sufficient to consider an additional impulsive term $(E^0_i - E^\star_i(0)) \cdot \delta(t)$ in the expression of $P^\star_i$. The following feedback law is introduced:

$$\varphi_i(E,t) = \begin{cases} \Pi(E_i, v_i) - f_i(E,t) & \text{if} \quad E_i > E_{MIN} \\ \Pi(E_{MIN}, v_i) & \text{if} \quad E_i = E_{MIN} \end{cases} \tag{21}$$

where the function f_i is obtained by differentiating with respect to time the expression in (17) when $i \in \mathscr{S}_1(t)$, evaluated at an arbitrary state E:

$$f_i(E,t) = \frac{d}{d\kappa}\Pi^{-1}_{E_i}\left(K\left(\sum_{i=1}^N E_i\right)\right)K'\left(\sum_{i=1}^N E_i\right)\left[\sum_{i=1}^N \Pi(E_i, v_i) - P_r(t)\right] \tag{22}$$

Proposition 3: If (20) holds for the initial state E^0, the optimal power profile $P^\star$ for (11) is equal to the feedback function φ evaluated along the optimal trajectory $E^\star$:

$$P^\star_i(t) = \varphi_i(E^\star(t), t) \qquad i = 1,\ldots,N. \tag{23}$$

Proof: It is sufficient to show that, given $E^\star_i$ defined in (17), its derivative corresponds to the dynamics (6) of the single turbine when $P^\star$ is applied. Notice that, for $E_i = E_{MIN}$, the feedback law is discontinuous. In this case, taking into account that $E^\star_i(\bar{t}) = E_{MIN}$ when $E^\star_i(t) = E_{MIN}$ and $\bar{t} > t$, the right derivative (equal to 0) can be considered. The following general expression can then be provided:

$$\begin{aligned}\frac{d}{dt}E^\star_i(t) &= f_i(E^\star(t),t) \cdot \operatorname{sign}(E^\star_i(t) - E_{MIN}) \\ &= \Pi(E^\star_i(t), v_i) - P^\star_i(t).\end{aligned} \tag{24}$$

The first equality holds by definition of f_i, the second one is obtained by replacing (21) in the expression (23) of $P^\star_i$. The proof is concluded by verifying that the last member in (24) is equal to (6) evaluated at $E = E^\star_i(t)$, $P_e = P^\star_i(t)$ and $v = v_i$. ■

From (17) and (24) one can conclude that the optimal scheduling, in case of unconstrained power, is achieved by controlling the turbines in two different ways. The kinetic energy of the generators with $E^\star_i(t) > E_{MIN}$ is reduced by imposing $\dot{E}^\star_i(t) = f_i(E^\star(t), t)$ so that the following holds:

$$\Pi_E(E^\star_i(t), v_i) = K(E^\star_{TOT}(t)) \qquad \forall i \in \mathscr{S}_1(t). \tag{25}$$

Once the i-th turbine reaches the minimum energy E_{MIN}, it remains in that state ($\dot{E}^\star_i(t) = 0$) and the energy reduction is performed with the same criterion on the remaining ones.

It is now possible to further discuss the choice of providing frequency response with a scheduling of the turbines which maximizes the total kinetic energy at the final time T.

Proposition 4: For a given initial state E^0, the solution of the maximization problem (11) guarantees feasibility of (10) for the largest class of aggregate power set-points P_r.

Proof: Consider an aggregate power reference $\tilde{P}_r : [0,T] \to \mathbb{R}_+$ for which (11) is unfeasible. Assuming $\tilde{P}_r$ is bounded, since the power of the single turbine is unconstrained, there exist $\tilde{t} < T$ defined as the maximum t such that (11) is feasible for $P_r = \tilde{P}_r$ restricted on the interval $[0,t]$. In the considered overproduction regime this implies $E^\star_1(\tilde{t}) = \cdots = E^\star_N(\tilde{t}) = E_{MIN}$ for the optimal states, with $\dot{E}^\star_{TOT}(\tilde{t}) < 0$. Take now an arbitrary power profile $\bar{P}$ with $\sum_i \bar{P}_i = \tilde{P}_r$. For the corresponding energy vector $\bar{E}$ it will hold $\sum_i \bar{E}_i(\tilde{t}) \leq \sum_i E^\star_i(\tilde{t}) = NE_{MIN}$. This means that $\bar{E}(\tilde{t}) = E^\star(\tilde{t})$ with $\dot{\bar{E}}_{TOT}(\tilde{t}) < 0$ or there exists at least one i such that $\bar{E}_i(\tilde{t}) < E_{MIN}$. We can conclude that there exists no power profile $P(\cdot)$ such that (10) is satisfied for $P_r = \tilde{P}_r$. ■

IV. Optimal Response with Constraints on Generated Power

The analysis is now extended in order to study the case of wind turbines that have a power output limited to the interval $[P_{MIN}, P_{MAX}]$. The corresponding control problem becomes:

$$\begin{aligned} \max_{P_i(\cdot), i=1\ldots N} \quad & \sum_{i=1}^N E_i(T) \\ \text{s. t.} \quad & \textstyle\sum_{i=1}^N P_i(t) = P_r(t) \\ & E_i(0) = E^0_i \\ & \dot{E}_i(t) = \Pi(E_i(t), v_i) - P_i(t) \qquad \begin{pmatrix} \forall i = 1,\ldots,N \\ \forall t \in [0,T] \end{pmatrix} \\ & E_i(t) \in [E_{MIN}, E_{MAX}] \\ & P_i(t) \in [P_{MIN}, P_{MAX}]. \end{aligned} \tag{26}$$

When applying the approach presented in the previous section, two important elements must be taken into account. The first one is the initialization of the turbines: if (20) is not satisfied, it follows from Remark 2 that the generators present different values of Π_E at $t = 0$. In this case, it is not possible to instantly correct such condition and impose equal derivatives by performing an energy switch with impulsive power. The second element to consider is that the power profile $P^\star_i$ defined in (23) and optimal for the unconstrained problem, albeit bounded for $t > 0$, may violate the power boundaries defined by P_{MIN} and P_{MAX}. In the rest of this section, on the basis of the insights provided by the analysis of the unconstrained problem, we present an heuristic control strategy that brings the turbines to equal levels of partial derivative $\Pi_E(E, v)$ and then preserves such equality for as long as possible. It is shown that the proposed choice solves the initialization issue of the turbines and, under some conditions, guarantees optimality.

A. Feedback Control Strategy for the Constrained Problem

In order to properly define our candidate solution for the optimization problem (26), some preliminary operations are required. In particular, given the vector $E \in [E_{MIN}, E_{MAX}]^N$ of kinetic energy values, we rank the generators for decreasing values of their partial derivative Π_E. This allows to introduce a permutation of the turbines population, denoting by $\tilde{i}(E)$ the turbine that occupies the i-th position in the derivative ordering for a given energy vector E. With this new indexing, the following chain of inequalities is satisfied:

$$\Pi_E(E_{\tilde{1}(E)}, v_{\tilde{1}(E)}) \geq \Pi_E(E_{\tilde{2}(E)}, v_{\tilde{2}(E)}) \geq \cdots \geq \Pi_E(E_{\tilde{N}(E)}, v_{\tilde{N}(E)}). \tag{27}$$

It is worth mentioning that such ordering always exists and is unique if one assumes, for example, that the original index order is preserved for turbines with equal derivatives. Equivalently, given $j_1 < j_2$ such that $\Pi_E(E_{j_1}, v_{j_1}) = \Pi_E(E_{j_2}, v_{j_2})$, for the corresponding new indexes $\tilde{i}_1(E)$ and $\tilde{i}_2(E)$ we have:

$$\tilde{i}_1(E) = j_1 \qquad \tilde{i}_2(E) = j_2 \qquad i_1 < i_2.$$

Definition 1: The proposed feedback control law $\phi(E,t) : [E_{MIN}, E_{MAX}]^N \times [0,T] \to [P_{MIN}, P_{MAX}]^N$ can be defined component-wise for decreasing values of i (starting from $i = N$) and has the following expression:

$$\phi_{\tilde{i}(E)}(E,t) = \begin{cases} \min\left(P_{MAX}, P_r(t) - \displaystyle\sum_{j=i+1}^{N} \phi_{\tilde{j}(E)}(E,t)\right) & \\ \qquad\qquad\qquad\qquad \text{if} \quad E_{\tilde{i}(E)} > E_{MIN} \\ \min\left(\Pi(E_{MIN}, v_{\tilde{i}(E)}), P_r(t) - \displaystyle\sum_{j=i+1}^{N} \phi_{\tilde{j}(E)}(E,t)\right) & \\ \qquad\qquad\qquad\qquad \text{if} \quad E_{\tilde{i}(E)} = E_{MIN}. \end{cases} \tag{28}$$

By setting $P(t) = \phi(E(t),t)$, the frequency response is provided by allocating maximum power on the turbine $\tilde{N}(E(t))$ which has the lowest partial derivative Π_E, taking into account that $P_{\tilde{N}(E)}(t) \leq P_{MAX}$ and $P_{\tilde{N}(E)}(t) \leq \Pi(E_{MIN}, v_{\tilde{N}(E)})$ if the turbine has reached the minimum energy level E_{MIN}. The same procedure is repeated for the $\tilde{i}(E(t))$-th turbine with $i = N-1$ and for decreasing value of i until the whole required power $P_r(t)$ has been allocated. Coherently with our initial premise, the turbines with lower values of Π_E will generate the maximum feasible amount of power, reducing their kinetic energy very rapidly and, consequentially, increasing their values of Π_E. The opposite occurs for the remaining generators, ensuring that the partial derivatives of the wind turbines are driven towards a common value.

Remark 3: The feedback law ϕ maximizes, at each time instant t, the derivative of the total mechanical power extracted from the wind, which has the following linear expression with respect to the P_is:

$$\frac{d}{dt}\sum_{i=1}^{N} \Pi(E_i(t), v_i) = \sum_{i=1}^{N} \Pi_E(E_i(t), v_i)\left(\Pi(E_i(t), v_i) - P_i(t)\right).$$

To verify this, it is sufficient to note that while the sum of the generated powers $P_i(t)$ is fixed and equal to $P_r(t)$, when ϕ is applied the maximum feasible power is allocated on the turbines with the lowest values of the positive quantity Π_E.

B. Optimality Results and Structure of the Solution

The feedback control law ϕ, presented in Definition 1 and designed heuristically on the basis of the analysis of the unconstrained case, is optimal for the optimization problem (26) with limited power generation if certain conditions are satisfied. To show this, we formally define the state trajectories and power profiles of the turbines when ϕ is applied:

Definition 2: Consider the following dynamical system, which describes the evolution of the turbines kinetic energies across time:

$$\begin{array}{rcl} \dot{E}_i(t) &=& \Pi(E_i(t), v_i) - P_i(t) \\ E_i(0) &=& E_i^0 \end{array} \qquad i = 1, \ldots, N. \tag{29}$$

Given the feedback control law ϕ presented in Definition 1, we denote by $E^* : [0,T] \to [E_{MIN}, E_{MAX}]^N$ and $P^* : [0,T] \to [P_{MIN}, P_{MAX}]^N$ the state trajectories and the control inputs of system (29) when the feedback ϕ is applied and $P_i(t) = \phi_i(E(t),t)$ for all $t \in [0,T]$ and $i = 1, \ldots, N$.

We initially assume that E^* and P^* always exist and are unique. Such assumption is discussed in the last part of this section, after Theorem 3. Next, we show that the feedback ϕ is optimal for the final energy maximization problem if the ordering of the turbines derivatives remains constant over time. Equivalently, the following condition must be satisfied.

$$\tilde{i}(E^*(t)) = \tilde{i} \qquad \forall t \in [0,T] \qquad i = 1, \ldots, N. \tag{30}$$

Theorem 3: Given E^* introduced in Definition 2, the final total energy $E^*_{TOT}(T) = \sum_{i=1}^{N} E_i^*(T)$ is the maximum of problem (26) if (30) holds.

Proof: See Appendix D ■

To fully understand the implications of Theorem 3 and determine when its hypothesis are verified, it is important to analyse the structure of the solutions of (29) when the feedback law ϕ is applied. To keep our analysis simple, we will consider the case $N = 2$ but the results can be easily extended to an arbitrarily large value of N. We assume without loss of generality that $\Pi_E(E_1^0, v_1) > \Pi_E(E_2^0, v_2)$ and, for simplicity, we consider $P_r(t) \in [P_{MAX}, 2 \cdot P_{MAX}]$ for all $t \in [0,T]$. If the minimum energy level E_{MIN} is never reached, the components of the feedback law ϕ have the following expression:

$$\phi_1(E,t) = \begin{cases} P_r(t) - P_{MAX} & \text{if} \quad \Pi_E(E_1, v_1) > \Pi_E(E_2, v_2) \\ P_{MAX} & \text{if} \quad \Pi_E(E_1, v_1) < \Pi_E(E_2, v_2) \end{cases}$$

$$\phi_2(E,t) = \begin{cases} P_{MAX} & \text{if} \quad \Pi_E(E_1, v_1) > \Pi_E(E_2, v_2) \\ P_r(t) - P_{MAX} & \text{if} \quad \Pi_E(E_1, v_1) < \Pi_E(E_2, v_2). \end{cases} \tag{31}$$

Three different possibilities, represented graphically in Fig. 4, must be considered for the state trajectories of (29). The first trivial case (A - blue trace in Fig. 4) is that $\Pi_E(E_1^*(t), v_1) > \Pi_E(E_2^*(t), v_2)$ for all $t \in [0,T]$. This means that the power generated by the two turbines is equal to $P_1^*(t) = P_r(t) - P_{MAX}$ and $P_2^*(t) = P_{MAX}$, respectively. The solution of system (29) can then be interpreted in a classical sense and it is straightforward to verify that (30) holds and the optimality of the control signal P^* is guaranteed. Alternatively to case A, there exists a time instant $\bar{t}$, defined as the minimum $t \in [0,T]$ such that $\Pi_E(E_1^*(t), v_1) = \Pi_E(E_2^*(t), v_2)$. In this

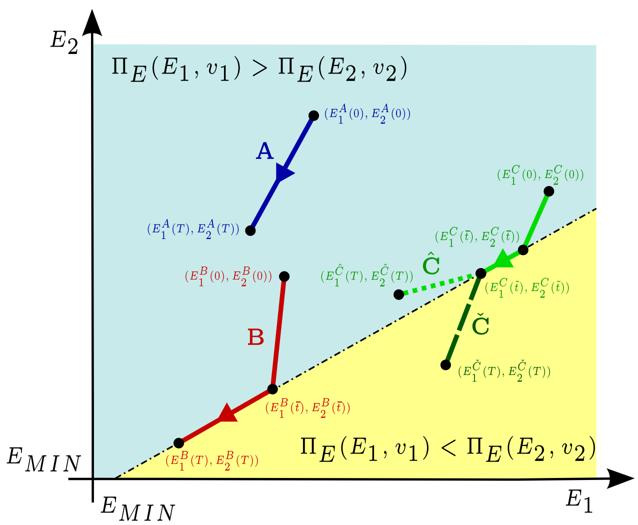

Fig. 4. State-space representation of different typologies of solutions for system (29) when $N = 2$ and the feedback control law ϕ is applied.

instance the system solution, for $t > \bar{t}$, could lie on the sliding surface characterized by $\Pi_E(E_1, v_1) = \Pi_E(E_2, v_2)$ (B - red trace in Fig. 4). From the results of the previous section, in case B the solution for $t \geq \bar{t}$ corresponds to the unconstrained optimal one (denoted by $(\bar{E}^\star, \bar{P}^\star)$) obtained solving (11) for $E_1^0 = E_1^*(\bar{t})$ and $E_2^0 = E_2^*(\bar{t})$. Adopting Filippov definition [13] for solutions of systems with discontinuous inputs, this is the case if the following conditions are satisfied:

$$\begin{aligned} \bar{P}_1^\star(t - \bar{t}) \in [P_r(t) - P_{MAX}, P_{MAX}] \\ \bar{P}_2^\star(t - \bar{t}) \in [P_r(t) - P_{MAX}, P_{MAX}] \end{aligned} \quad \forall t \in [\bar{t}, T]. \tag{32}$$

Remark 4: Two important considerations can be made if (32) holds and the solution of (29) lies on the sliding surface. From a practical point of view, the implementation of the feedback law ϕ does not require high frequency switching of the control inputs when the turbines reach equal values of the derivative Π_E. In this case one can simply apply the power profiles $\bar{P}^\star$ obtained by solving the unconstrained problem. It can also be proven that $P^*(t) = \phi(E^*(t), t)$ is still optimal for (26). In fact, as a result of Theorem 3, the total kinetic energy is maximized over $[0, \bar{t}]$. Moreover, the state trajectory on $[\bar{t}, T]$ is equal to the one of the unconstrained problem, whose objective function coincides with the original one and is monotonic increasing with respect to the total energy $E_1^*(\bar{t}) + E_2^*(\bar{t})$.

The last possibility to consider (C - green trace in Fig. 4) is that (32) does not hold for some $\tilde{t} \in [\bar{t}, T]$ and the state trajectory leaves the sliding surface. If it returns to the region of the state-space characterized by $\Pi_E(E_1, v_1) > \Pi_E(E_2, v_2)$ (Ĉ - dotted green trace in Fig. 4), the initial ordering of the derivatives Π_E is preserved and the optimality results discussed for the previous scenarios still hold. If, on the other hand, the state trajectory "crosses" the sliding surface (Č - dashed green trace in Fig. 4) the original derivative ordering is altered. The scenario Č is the only one for which it is not possible to prove the optimality of ϕ. On the other hand, one can still apply Theorem 3 and previous considerations to the time subintervals $[0, \tilde{t}]$ and $[\tilde{t}, T]$. In particular, the total kinetic energy is maximized at $\tilde{t}$ and, for the resulting energy state $E^*(\tilde{t})$, also the final kinetic energy is maximized.

Remark 5: It is worth mentioning that, in a scenario with wind turbines affected by equal wind speed, case C never occurs. In fact, in this instance, we have $P_1^\star(t) = P_2^\star(t)$ when $\Pi_E(E_1^\star(t), v_1) = \Pi_E(E_2^\star(t), v_2)$. This means that (32) always holds and the power profile P^* obtained with the feedback ϕ is optimal for the frequency response problem (26).

V. Simulation Results

The performance of the proposed scheduling has been evaluated in simulations. The turbine parameters presented in [10] have been adopted, converting the operative interval of the rotor speed to the corresponding kinetic energy values:

$$\begin{aligned} R &= 37.5m & J &= 5.9 \cdot 10^6 Kg \cdot m^2 \\ P_{MIN} &= 0MW & P_{MAX} &= 2MW \\ E_{MIN} &= 2.62 \cdot 10^6 J & E_{MAX} &= 1.43 \cdot 10^7 J. \end{aligned} \tag{33}$$

A qualitative representation of the considered scenario is presented in Fig. 5: initially all generators are operating at the point of maximum efficiency, generating the aggregate power P_{ss} specified in (9). At time $t = 0$, supposing a frequency event occurs, the reference for the aggregate power is increased:

$$P_r(t) = 1.3 \sum_{i=1}^{N} \Pi(E_{ss}(v_i), v_i) = 1.3 P_{ss}. \tag{34}$$

The power profiles of the turbines are calculated in order to achieve a total generated power P_{TOT} which is equal to P_r at each time instant, as specified by the constraints in (11) and (26), minimizing at the same time the total energy losses. After having provided frequency response, the turbines move to a recovery phase: their generated power is reduced so they can increase their kinetic energy and move back to the operating point of maximum efficiency which characterizes normal operation. The recovery problem is studied in detail in Section VI.

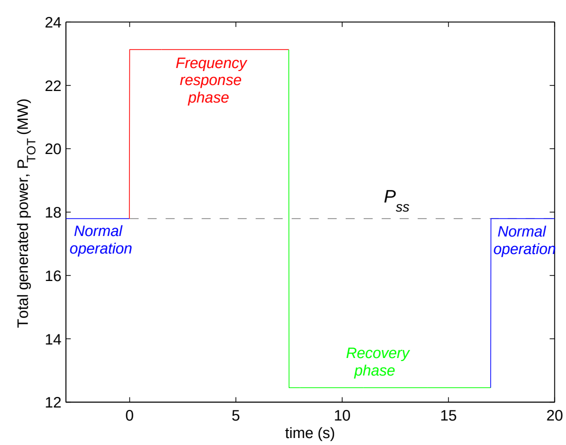

Fig. 5. Total power generated by the wind farm at different operational modes.

Remark 6: In practical applications the proposed scheduling can be implemented in real time with a centralized approach that requires a two-way communication channel between the turbines and a central entity. The latter receives the measured wind speeds and current kinetic energies from the generators and calculates, at each time t, the power required from each turbine. In the unconstrained case it solves (12), determining the resulting power through (23). For the constrained case, the feedback ϕ presented in Definition 1 is applied. The required power values are communicated to each turbine and converted to the equivalent torque reference $T_{e_{ref}}$ which constitutes the input of the electrical control loop shown in Fig. 1.

A. Optimal Scheduling in the Unconstrained Case

A first analysis focuses on the case of turbines with unconstrained power, considering a population of $N = 20$ generators. Initially, they are all operating at the point of maximum efficiency $E_i(0) = E_{ss}(v_i)$ with different wind speeds in the interval $[8m/s, 10m/s]$ and equal derivatives $\Pi_E(E_1(0), v_1) = \cdots = \Pi_E(E_N(0), v_N) = 0$. The optimal scheduling is calculated with a time step $\Delta t = 0.05s$, solving the static optimization problem (12) at each time instant $l \cdot \Delta t$. The optimal energy trajectories $E^\star$ for each turbine are shown in Fig. 6. Notice that, after the frequency event occurs at $t = 0$, the kinetic energy is reduced across time in all generators in order to maintain an equal derivative Π_E. The corresponding optimal power profiles $P^\star$ are shown in Fig. 7: after the initial increase, the generated power is approximately constant for all turbines. When the slowest turbine (let it be turbine i) reaches the minimum energy E_{MIN}, its power generation is instantaneously reduced to $\Pi(E_{MIN}, v_i)$ and the control effort is redistributed among the remaining generators which, as a consequence, increase their individual power output considerably in order to meet overall power requirements. Since turbines have equal derivatives Π_E at time $t = 0$ and therefore satisfy (20), there is no impulsive energy switch. The optimal power in the unconstrained case is finite and in this case it is also within the operational limit $P_{MIN} < P < P_{MAX}$ of the turbines except for the very last part of the frequency response when the power output of some turbines is indirectly limited by the fact that they have reached the minimum feasible energy E_{MIN}.

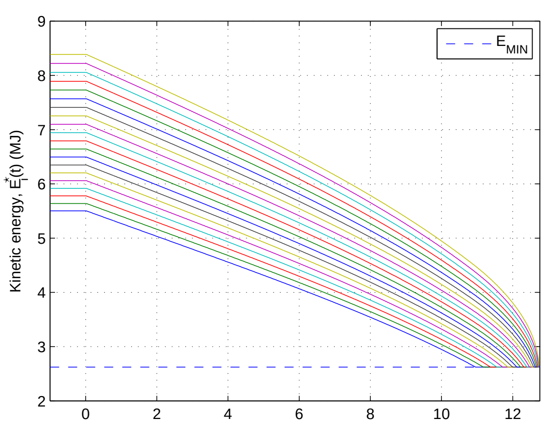

Fig. 6. Kinetic energy of the individual wind turbines when providing frequency response in the unconstrained power case.

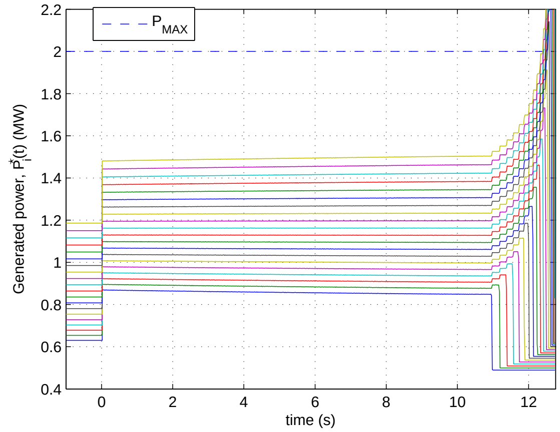

Fig. 7. Optimal power profiles $P_i^\star$ for scheduling in the unconstrained case.

B. Scheduling of Turbines with Limited Power

A similar scenario with constraints on the generated power P is now simulated: in this case it is assumed that turbines have different derivative Π_E at $t = 0$, with wind speed and initial state defined as follows for $j = 1, \ldots, 20$:

$$v_j = 8 + 0.1 \cdot j \qquad E_j(0) = E_{ss}(v_j) \cdot (1.02 - 0.02 \cdot j). \quad (35)$$

The frequency response is provided with the heuristic control strategy introduced in Section IV, applying the feedback law ϕ presented in Definition 1. The resulting state trajectories $E_i^*(t)$, generated power profiles $P_i^*(t)$ and partial derivatives $\Pi_E(E_i^*(t), v_i)$ of the individual generators, with $i = 1, \ldots, N$, are shown, respectively, in Fig. 8, 9 and 10. Three different time intervals can be analysed separately, on the basis of the values of Π_E. At the beginning of the frequency response (from 0 to 1 second) the power derivatives are distinct: the aggregate power P_r is allocated by setting $P_i^*(t) = P_{MAX}$ for the turbines that have lower Π_E which, as a result, are slowed down. Since P_r is not an integer multiple of P_{MAX}, one of these turbines will generate a power which is lower than the maximum. It can be seen in Fig. 10 that in the first second the generators converge one after another to equal values of Π_E. In the considered scenario, every time this happens, the energy trajectories of the turbines will be similar to case B in Fig. 4. In fact, the equivalent of condition (32) is satisfied for the group of turbines with equal derivative, implying that the corresponding solution of the unconstrained problem is feasible. One turbine moves from maximum (or minimum) generation to some intermediate value and the power of the other generators is adjusted accordingly (see Fig. 9). The result is that the energy trajectories will lie on the sliding surface characterized by equal values of Π_E, as it can be inferred from Fig. 10. In a second time interval (approximately from 1 to 6 seconds), all turbines have reached the same value of Π_E and proceed on the sliding surface. The kinetic energy is gradually reduced, the common value of Π_E increases over time and the power P_i^* of each generator remains almost constant. In the last interval, when turbines start reaching the minimum energy value E_{MIN}, their power is reduced to $P_i^*(t) = \Pi(E_{MIN}, v_i)$

and such variation is compensated by increasing the power on the remaining ones until feasibility can be guaranteed. It is worth mentioning that, in the presented scenario, the initial ordering of the derivatives Π_E is preserved throughout the whole considered time interval $[0,T]$. From Theorem 3 and considerations of the previous section, this implies that in the present case the feedback ϕ is optimal for problem (26).

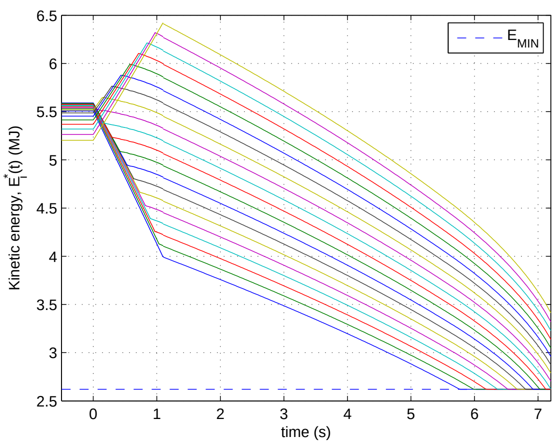

Fig. 8. Kinetic energy of the individual wind turbines when providing frequency response in the constrained power case.

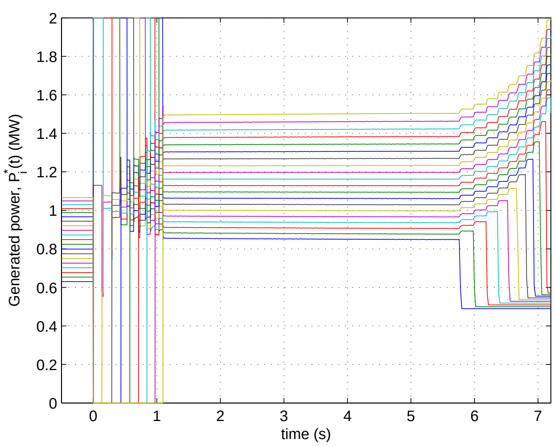

Fig. 9. Power profiles P_i^* of the single generators for scheduling with constraints on generated power.

The results obtained with the discussed turbine scheduling are compared to the ones of a traditional approach, where each turbine increases proportionally its generated power in order to meet the same overall power requirements. The total kinetic energy of the wind farm in the two cases, for the parameters mentioned above, is denoted respectively by E^*_{TOT} and $\bar{E}_{TOT}$ and is shown in Fig. 11. As expected, the aggregate energy of the turbines is higher with the proposed scheduling. This means that the subsequent recovery phase, when turbines generate less power to recover their kinetic energy, will require a smaller counterbalance from other sources of generation and therefore will have a reduced negative impact on the system.

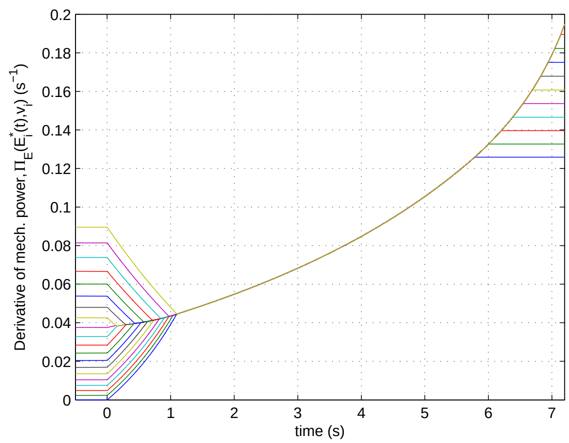

Fig. 10. Partial derivative $\Pi_E(E_i^*(t),v_i)$ of the turbines for scheduling with constrained power.

Moreover, the curves in Fig. 11 stop at different time instants, corresponding to the points at which the turbines are not able to provide the required amount of aggregate power under the specified energy and power constraints. With the proposed control strategy, it is possible to provide a frequency response which is 15% longer than the standard one. Consequently, there is more time for the secondary response to kick in and correct the power imbalance in the system, allowing the use of cheaper generators with lower ramping rates and achieving significant economical benefits. Finally, for the chosen values of initial energy and wind speed, it is supposed that a certain time T of frequency response is required from the turbines. The maximum percentage of power increase ΔP which can be delivered with the proposed scheduling by setting $P_r(t) = P_{ss}(1+\Delta P)$ for all $t \in [0,T]$ is shown in Fig. 12. At lower values of T (and higher power increase factors), the reduction of ΔP is more significant. In this case the turbines are generating more power, moving away faster from their initial operating point and introducing a larger reduction of efficiency.

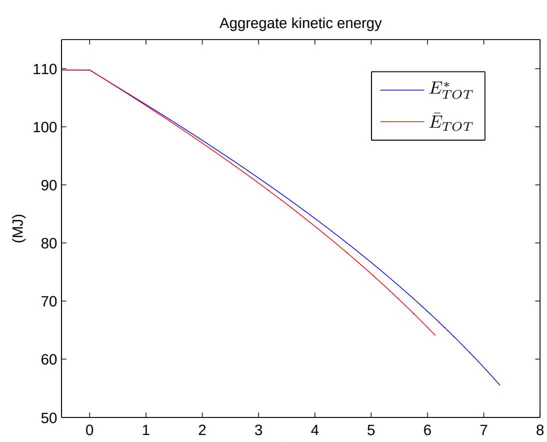

Fig. 11. Total kinetic energy of the wind farm after a frequency event with the proposed scheduling (E^*_{TOT}) and with traditional techniques, applying a proportional increase of power generation equal for all turbines ($\bar{E}_{TOT}$).

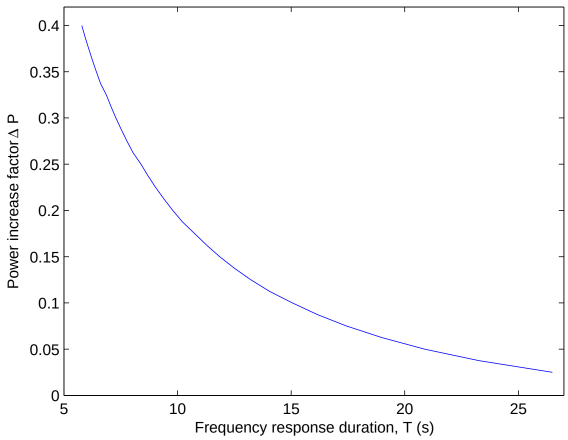

Fig. 12. Maximum percentage increase ΔP of total generation achievable with the proposed scheduling when T seconds of frequency response are required.

VI. Optimal Energy Recovery

When turbines provide frequency response they release part of their kinetic energy, moving away from their operative point of maximum efficiency. Therefore, after the frequency support phase, it is desirable to bring back the turbines to their optimal rotor speed (and corresponding kinetic energy). The recovery is approached as an optimal control problem, imposing a minimum value P_L of aggregate power and calculating the power profiles P_i which minimize the time required to restore the original configuration of maximum production:

$$\begin{array}{ll} \displaystyle\min_{T,P_i(\cdot),i=1\ldots N} & T \\ \text{s. t.} & \sum_{i=1}^N P_i(t) \geq P_L \\ & E_i(0) = E_i^0 \\ & E_i(T) = E_{ss}(v_i) \\ & \dot{E}_i(t) = \Pi(E_i(t),v_i) - P_i(t) \\ & E_i(t) \in [E_{MIN}, E_{ss}(v_i)] \\ & P_i(t) \in [P_{MIN}, P_{MAX}]. \end{array} \quad \left(\begin{array}{c} \forall i = 1,\ldots,N \\ \forall t \in [0,T] \end{array}\right) \tag{36}$$

The feasibility of the problem is initially assessed, determining the values of P_L in the interval $[NP_{MIN}, NP_{MAX}]$ for which a solution exists. A preliminary assumption is made for the constraints on the generated power P_i of the single turbine imposing that, for any feasible value E_i, it is always possible to increase or reduce the kinetic energy of the turbine:

$$0 \leq P_{MIN} < \Pi(E_i, v_i) < P_{MAX} \qquad \begin{array}{c} i = 1,\ldots,N \\ E_i \in [E_{MIN}, E_{ss}(v_i)]. \end{array} \tag{37}$$

A first feasibility result can now be provided with the following sufficient condition:

Proposition 5: Given the initial state E^0, the value P_L for the minimum total generated power is feasible for problem (36) if:

$$\Pi_{TOT}\left(E^0\right) = \sum_{i=1}^N \Pi\left(E_i^0, v_i\right) > P_L \tag{38}$$

where Π_{TOT} denotes the aggregate mechanical power extracted by the wind farm.

Proof: See Appendix E. ■

If the sufficient condition (38) does not hold, there is no power profile which allows to instantly increase the energies E_i of all turbines. It is still possible, on the other hand, to provide milder conditions for the feasibility of the recovery problem, which in this case are necessary and sufficient:

Proposition 6: For a given initial state E^0 and minimum aggregate generation P_L, problem (36) is feasible if and only if there exists a time $\tau \geq 0$ and a power profile $\bar{P}(\cdot)$ such that, for the corresponding energy vector $\bar{E}$, it holds:

$$\Pi_{TOT}(\bar{E}(\tau)) = \sum_{i=1}^N \Pi(\bar{E}_i(\tau), v_i) > P_L. \tag{39}$$

Proof: If τ specified in the claim does not exist, it follows that the derivative of the total energy stored in the turbines is always negative and therefore, since $E_i(0) < E_{ss}(v_i)$ $\forall i$, problem (36) is infeasible. If, on the other hand, (39) is satisfied, the feasibility is guaranteed by Proposition 5, considering $E^0 = \bar{E}(\tau)$ as the initial energy vector. ■

This means that the feasibility of problem (36) can be determined by solving the following problem for increasing values of τ and comparing its solution with P_L:

$$\begin{array}{ll} \displaystyle\max_{P_i(\cdot),i=1\ldots N} & \Pi_{TOT}(E(\tau)) \\ \text{s. t.} & \sum_{i=1}^N P_i(t) \geq P_L \\ & E_i(0) = E_i^0 \\ & \dot{E}_i(t) = \Pi(E_i(t),v_i) - P_i(t) \\ & E_i(t) \in [E_{MIN}, E_{ss}(v_i)] \\ & P_i(t) \in [P_{MIN}, P_{MAX}]. \end{array} \quad \left(\begin{array}{c} \forall i = 1,\ldots,N \\ \forall t \in [0,\tau] \end{array}\right) \tag{40}$$

One can extend previous optimization results and prove that under certain conditions the feedback ϕ, presented in Definition 1, is optimal for (40) and can therefore be used to analyse the feasibility of (36) for the considered value of P_L.

Proposition 7: Consider the power profile P^* and the state trajectory E^* introduced in Definition 2, assuming $P_r(t) = P_L$ and $T = \tau$. If the condition of constant indexing (30) holds, P^* is optimal for problem (40).

Proof: From Theorem 3, under the current assumptions, P^* is optimal and therefore also feasible for problem (26) with $P_r(t) = P_L$ and $T = \tau$. The only difference to be considered between the constraints of (26) and (40) is given by the stricter constraint $E_i(t) \leq E_{ss}(v_i)$ in (40). Since this is never violated when ϕ is applied (if $E_i(t) = E_{ss}(v_i)$ the power generated by the i-th turbine is $P_i^*(t) = P_{MAX} > \Pi(E_{ss}(v_i), v_i)$), we can conclude that P^* is feasible for (40). Consider now the change of coordinates introduced in the proof of Theorem 3 and described by equations (50) and (51). The objective function of (40) can alternatively be rewritten as:

$$\Pi_{TOT}(E(\tau)) = \sum_{i=1}^N \Pi(E_i(\tau), v_i) = \dot{\tilde{E}}_N + P_r(\tau) = \dot{\tilde{E}}_N + P_L$$

which from (52) has positive partial derivatives with respect to all the state components $\tilde{E}_i$ with $i = 1\ldots,N$. From the proof of Theorem 3, under the current assumptions, such components are maximized at final time $T = \tau$ by P^*. Therefore, P^* is optimal also for problem (40). ■

The results provided in Section IV can also be used to solve the current problem of time minimization:

Theorem 4: Consider the power profile P^* and state trajectory E^* introduced in Definition 2, assuming $P_r(t) = P_L$ and $T = T^*$. If the condition of constant indexing (30) holds and the constraints of (36) are satisfied for $P = P^*$ and $T = T^*$, then (P^*, T^*) is an optimal solution of (36).

Proof: From the proof of Theorem 3 we can conclude that, under the current assumptions, P^* is optimal for (26) with $P_r(t) = P_L$ and $T \in [0, T^*]$. In fact, P^* maximizes at each time instant the state derivatives of the monotone dynamical system described by (50) and (51) and the objective function of (26) is equal to the state component $\tilde{E}_N(T)$. Since we are assuming that P^* is feasible for (36), the following holds for the resulting kinetic energy E^* of the turbines:

$$\sum_{i=1}^{N} E_i^*(t) < \sum_{i=1}^{N} E_i^*(T^*) = \sum_{i=1}^{N} E_{ss}(v_i) \quad \forall t \in [0, T^*). \tag{41}$$

If P^* is not optimal for (36), there exists a power profile $\bar{P}$ and the corresponding energy vector $\bar{E}$ such that, for $T = \bar{T} < T^*$, we have $\bar{E}_i(\bar{T}) = E_{ss}(v_i)$ with $i = 1, \dots, N$. As a result, the following must hold at $t = \bar{T}$:

$$\sum_{i=1}^{N} \bar{E}_i(\bar{T}) = \sum_{i=1}^{N} E_{ss}(v_i) > \sum_{i=1}^{N} E_i^*(\bar{T}) \tag{42}$$

but this contradicts the optimality of P^* for problem (26) with final time $\bar{T} \in [0, T]$. ■

We can conclude that the recovery of the turbines is structurally similar to the frequency response problem with limited power. The heuristic control strategy described by the feedback law ϕ represents a reasonable choice to bring back the turbines to their original working point with maximum efficiency. In fact, from Remark 3, ϕ maximizes at each time instants the derivative of the total mechanical power absorbed by the turbines. Moreover, if certain conditions are satisfied, ϕ can be used to test which values of minimum aggregate generation are feasible (Proposition 7) and to obtain recovery in minimum time (Theorem 4).

The feasibility conditions and the scheduling for the energy recovery problem have been tested in simulations adopting the parameters (33) used in Section V. In particular, the feasibility conditions presented in Proposition 5 and 6 have been applied to the simple case of $N = 3$ turbines with different wind speeds $v = [8m/s, 9m/s, 10m/s]$ and minimum aggregate power $P_L = 2.5MW$. The boundaries of the feasibility regions with respect to the initial energy E_i^0 of each turbine are shown in Fig. 13. The border B_1 (in red) delimits the initial energy values for which a recovery scheduling exists (from Proposition 6) while the border B_2 (in blue) denotes the smaller area, defined by Proposition 5, in which is possible to initially accelerate all turbines at the same time.

The proposed scheduling is also compared with the recovery of the turbines when a standard optimum power point tracking (OPPT) is used and the power generated by the i-th turbine is defined as a function of the rotor speed ω [14]:

$$\bar{P}_i(\omega_i) = \frac{\mu \pi R^5 \bar{C}(\lambda_{ss}(0), 0)}{2\lambda_{ss}^3(0)} \omega_i^3 = K_T \omega_i^3 \tag{43}$$

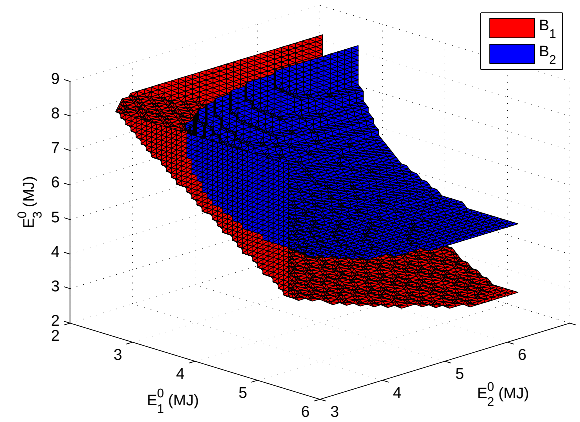

Fig. 13. Borders of the feasibility region of the recovery problem with respect to the initial state E^0, as defined by Proposition 5 (blue) and Proposition 6 (red) in the case of $N = 3$ turbines.

where $\lambda_{ss}(0)$ denotes the tip-speed ratio which maximizes the power coefficient $\bar{C}(\lambda, \theta)$ when $\theta = 0$. The aggregate power profile P_{TOT} generated with this controller is then used as reference for the recovery problem (36), extended to consider time-varying P_L. The kinetic energies of the turbines in the two cases are compared in Fig. 14. Notice that, when the feedback ϕ is applied, only some turbines are initially accelerated and then, once equal values of Π_E are obtained, the equality of the partial derivatives is preserved. The total kinetic energy of the turbines in the two cases (respectively $\bar{E}_{TOT}$ and E^*_{TOT}) is shown in Fig. 15: as expected, the proposed scheduling is able to achieve, for the same aggregate generated power, a faster recovery of the wind farm.

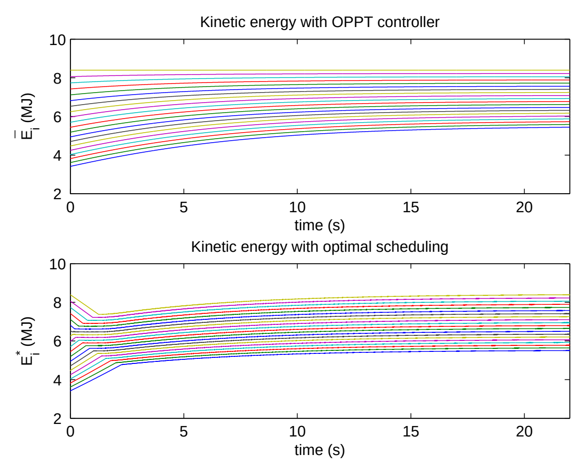

Fig. 14. Kinetic energy of each wind turbine during recovery when a standard OPPT controller is used (top) and when the scheduling with feedback ϕ is applied (bottom).

VII. Conclusions

A new methodology is presented for the scheduling of wind turbines that provide frequency response, considering

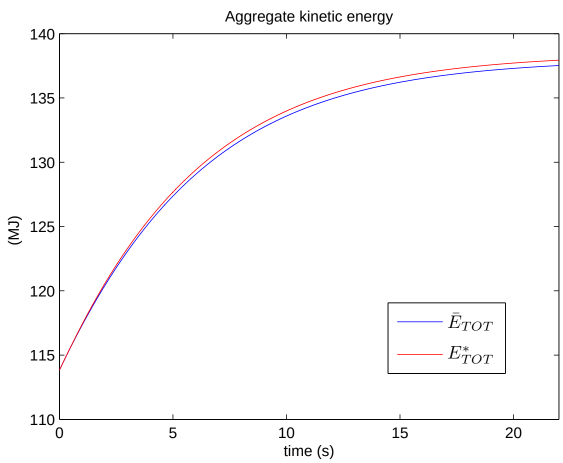

Fig. 15. Comparison between the aggregate energy $\bar{E}_{TOT}$ obtained with standard OPPT controller (blue) and the total energy E^*_{TOT} resulting from the application of the scheduling with feedback ϕ (red).

wind speed and angular velocity different in general for each turbine. In case of frequency events, the aggregate generation is increased by a specified amount, determining the power profile of the individual turbines through the resolution of an optimal control problem which minimizes energy losses. The proposed technique provides an upper bound on the efficiency of wind farms in these scenarios and constitutes a theoretical framework that can be expanded to include provisions of other ancillary services. The results are evaluated through simulations and the approach is also extended to the energy recovery problem, bringing back the turbines to the initial state of maximum efficiency in minimum time.

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Appendix A
Proof of Theorem 1

The feasibility of $x^\star$ is straightforward to verify: the constraint (12b) is satisfied from Proposition 1 while for (12c) it is sufficient to notice that, for the considered values of E_{TOT}, the value returned by $\Pi_{E_i}^{-1}$ in (15) is always in the interval $[E_{MIN}, E_{ss}(v_i)]$, with $E_{ss}(v_i) < E_{MAX}$ from Assumption 3. The optimality of the candidate solution is now proved through Karush-Kuhn-Tucker (KKT) conditions. In this particular case such conditions are necessary and sufficient since the inequality constraints are convex, equation (12b) is affine and the objective function is strictly concave [15]. To show this last point, consider that the Hessian of the objective function $H = \mathrm{diag}(\Pi_{EE}(x_1,v_1),\ldots,\Pi_{EE}(x_N,v_N))$ is negative definite since the second derivative $\Pi_{EE}(E,v) = \partial^2\Pi(E,v)/\partial E^2$ is negative from the strict concavity of Π established in Assumption 3. For the proposed solution the constraint on the maximum energy E_{MAX} is never active (the corresponding multiplier will always be equal to 0) and only the inequality in the opposite sense $x_i \geq E_{MIN}$ must be considered when deriving the KKT conditions. Therefore, the vector $x^\star$ is optimal if and only if there exists κ and $\mu = [\mu_1,\ldots,\mu_N]$ such that:

$$\begin{aligned} \Pi_E(x_i^\star, v_i) &= -\mu_i + \kappa \\ \mu_i &\geq 0 \qquad\qquad i = 1,\ldots,N. \\ \mu_i \cdot (E_{MIN} - x_i^\star) &= 0 \end{aligned} \tag{44}$$

These conditions are satisfied if one chooses the multipliers as follows:

$$\kappa = K(E_{TOT})$$
$$\mu_i = \begin{cases} 0 & \text{if} \quad \Pi_E(E_{MIN}, v_i) > \kappa \\ \kappa - \Pi_E(E_{MIN}, v_i) & \text{if} \quad \Pi_E(E_{MIN}, v_i) \leq \kappa. \end{cases} \tag{45}$$

Uniqueness of the optimal solution follows from its existence and the strict concavity of the function to maximize.

Appendix B
Proof of Proposition 2

Consider that the right hand side in (12a) is a strictly concave function of x and the compact-valued continuous correspondence $D(E_{TOT})$ which returns the set of feasible x for a given E_{TOT} has a convex graph. It is therefore possible to apply the maximum theorem [16, Theorem 9.17], considering E_{TOT} as a parameter of the optimization and concluding that h is strictly concave with respect to E_{TOT} on $[NE_{MIN}, \sum_{i=1}^{N} E_{ss}(v_i)]$. Following the concavity properties introduced in Assumption 3, the definition of the derivative inverse $\Pi_{E_i}^{-1}$ can be extended to the interval $[\Pi_E(E_{MAX}, v_i), \Pi_E(E_L(v_i), v_i)]$. It is also possible to define the domain of h as the interval $\mathscr{E}_D = \left[\sum_{i=1}^{N} E_L(v_i), \sum_{i=1}^{N} E_R(v_i)\right]$ which satisfies the following property:

$$\left[NE_{MIN}, \sum\nolimits_{i=1}^{N} E_{ss}(v_i)\right] \subset \mathscr{E}_D \subset \left[\sum\nolimits_{i=1}^{N} E_L(v_i), NE_{MAX}\right] \quad (46)$$

where $E_R(v_i) < E_{MAX}$ is the maximum value for the optimal energy of the i-th turbine for which it holds:

$$\Pi_E(E_R(v_1), v_1) = \cdots = \Pi_E(E_R(v_N), v_N) < 0. \quad (47)$$

Such domain definition is possible if one considers that in each $[E_L(v_i), E_R(v_i)]$ the monotonicity and concavity properties mentioned in the previous remarks still hold. We can then extend the results of Proposition 1 and Theorem 1, repeating the steps of the corresponding proofs for the new interval $\mathscr{E}_D$. With the same reasoning detailed at the beginning of this proof, applying the maximum theorem, we can show that h is concave with respect to E_{TOT} on $\mathscr{E}_D$. Therefore, h is also Lipschitz continuous in the same variable on the interval $[NE_{MIN}, \sum_{i=1}^{N} E_{ss}(v_i)] \subset \mathscr{E}_D$.

Appendix C
Proof of Theorem 2

For the feasibility of $E^\star$ notice that, at any time instant $t \in [0, T]$, it holds:

$$\begin{aligned} \dot{E}^\star_{TOT}(t) &= h(E^\star_{TOT}(t)) - P_r(t) \\ &= \left(\sum_{i=1}^{N} \Pi(E_i^\star(t), v_i)\right) - P_r(t) = \sum_{i=1}^{N} \dot{E}_i^\star(t). \end{aligned}$$

Taking into account that individual generated power P_is are unconstrained, it is always possible to determine input profiles $P_i^\star(t)$, with $i = 1, \dots, N$, which satisfy $\sum_{i=1}^{N} P_i^\star(t) = P_r(t)$ and the constraint on the state derivative $\dot{E}_i^\star(t) = \Pi(E_i^\star(t), v_i) - P_i^\star(t)$. Consider now an arbitrary state trajectory $\bar{E}(\cdot)$ which is feasible for (11) and define, at each time instant t, the corresponding total energy $\bar{E}_{TOT}(t) = \sum_{i=1}^{N} \bar{E}(t)$. Such function is differentiable since it holds:

$$\dot{\bar{E}}_{TOT}(t) = \sum_{i=1}^{N} \Pi(\bar{E}_i(t), v_i) - P_r(t). \quad (48)$$

From the definition of h, it follows:

$$\dot{\bar{E}}_{TOT}(t) \leq h(\bar{E}_{TOT}(t)) - P_r(t) \;\; \forall t \in [0, T]. \quad (49)$$

Considering that $\bar{E}_{TOT}(0) = E^\star_{TOT}(0)$ and applying the comparison theorem [17, Theorem 7], we can conclude that $\bar{E}_{TOT}(t) \leq E^\star_{TOT}(t)$ for all $t \geq 0$ including $t = T$ and therefore $E^\star(\cdot)$ is optimal for problem (11).

Appendix D
Proof of Theorem 3

Notice that, when (30) holds, each individual turbine is represented by the same index $\tilde{i}(E^*(t)) = \tilde{i}$ over the whole time interval $[0, T]$ and therefore occupies always the i-th place in the ordering with respect to Π_E. We now introduce the following change of coordinates for system (29):

$$\tilde{E}_i = \sum_{j=1}^{i} E_{\tilde{j}} \qquad \tilde{P}_i = P_{\tilde{i}} \quad (50)$$

where $\tilde{E}$ and $\tilde{P}$ represent, respectively, states and inputs of the dynamical system in the new coordinates. Accordingly, the state derivatives can be expressed as:

$$\dot{\tilde{E}}_i = \sum_{j=1}^{i} \Pi(E_{\tilde{j}}, v_{\tilde{j}}) - \sum_{j=1}^{i} P_{\tilde{j}} = \sum_{j=1}^{i} \Pi(\tilde{E}_j - \tilde{E}_{j-1}, v_{\tilde{j}}) - \sum_{j=1}^{i} \tilde{P}_j. \quad (51)$$

One can verify that the system described by (50) and (51) is monotone for the orders induced from orthant $\mathbb{R}^N_{\geq 0}$ for the state $\tilde{E}$ and $\mathbb{R}^N_{\leq 0}$ for the control $\tilde{P}$. To prove this, we apply [18, Corollary III.3] and show that the following inequalities are satisfied:

$$\frac{\partial \dot{\tilde{E}}_i}{\partial \tilde{E}_j} \geq 0 \qquad \frac{\partial \dot{\tilde{E}}_i}{\partial \tilde{P}_j} \leq 0 \qquad \begin{array}{l} i = 1, \dots, N \\ j = 1, \dots, N. \end{array} \quad (52)$$

It is straightforward to verify from (51) that the partial derivatives with respect to $\tilde{P}_j$ are never positive. For the case of derivation with respect to $\tilde{E}_j$ the only non trivial case is $1 \leq j < i$, for which we have:

$$\begin{aligned} \frac{\partial \dot{\tilde{E}}_i}{\partial \tilde{E}_j} &= \Pi_E(\tilde{E}_j - \tilde{E}_{j-1}, v_{\tilde{j}}) - \Pi_E(\tilde{E}_{j+1} - \tilde{E}_j, v_{(\widetilde{j+1})}) \\ &= \Pi_E(E_{\tilde{j}}, v_{\tilde{j}}) - \Pi_E(E_{(\widetilde{j+1})}, v_{(\widetilde{j+1})}) \geq 0 \end{aligned}$$

where the inequality holds as a result of (27) and condition (30) of constant indexing assumed in the theorem statement. Having established the monotonicity of the dynamical system described by (50) and (51), we denote by $\mathscr{P}$ the set of power profiles $P : [0, T] \to [P_{MIN}, P_{MAX}]^N$ that are feasible for (26) and remind that P^* is obtained by applying the feedback law ϕ, as specified in Definition 2. From expression (28) of ϕ, it is straightforward to verify that $P^* \in \mathscr{P}$. Moreover, when ϕ and the corresponding $P = P^*$ are applied, all state derivatives $\dot{\tilde{E}}_i(t)$ are maximized over the set of feasible profiles $\mathscr{P}$ at all $t \in [0, T]$ and $i \in \{1, \dots, N\}$. To see this, notice from (51) that $P_{\tilde{j}}(t)$ appears with negative sign in all derivatives $\dot{\tilde{E}}_i(t)$ with $i \geq j$ and the total sum $\sum_{j=1}^{N} P_{\tilde{j}}(t)$ is fixed and equal to $P_r(t)$. This means that all $\dot{\tilde{E}}_i(t)$ are maximized by allocating maximum feasible power on the turbines $\tilde{j}$ with highest j or, in other words, by applying ϕ. Therefore, from the monotonicity of the dynamical system in the new coordinates, each state component $\tilde{E}_j(t)$ is maximized by ϕ at all $t \in [0, T]$. The proof is concluded by noticing that $\tilde{E}_N(T) = \sum_{j=1}^{N} E_{\tilde{j}}(T)$ corresponds to the objective function of problem (26) and therefore $\tilde{E}_N^*(T) = E^*_{TOT}(T)$ is optimal for (26), as claimed in the theorem statement.

APPENDIX E
PROOF OF PROPOSITION 5

We show that, if (38) holds, there exists at least one power profile $\bar{P}(\cdot)$ that satisfies the constraints in (36). In order to construct $\bar{P}$, denote the set of turbines that have reached the desired terminal state by $\mathscr{F}(E) := \{i : E_i = E_{ss}(v_i)\}$. The power profile can now be defined through the following feedback law for $i = 1, \dots N$:

$$\bar{\varphi}_i(E) = \begin{cases} \Pi(E_{ss}(v_i), v_i) & i \in \mathscr{F}(E) \\ \max\left(\Pi(E_i, v_i) \cdot r(E), P_{MIN}\right) & i \notin \mathscr{F}(E) \end{cases} \tag{53}$$

where the function $r(E)$ is defined as follows:

$$\begin{aligned} r(E) &= \frac{P_L - \sum_{i\in\mathscr{F}(E)} \Pi(E_{ss}(v_i), v_i)}{\Pi_{TOT}(E) - \sum_{i\in\mathscr{F}(E)} \Pi(E_{ss}(v_i), v_i)} \\ &= \frac{P_L - \sum_{i\in\mathscr{F}(E)} \Pi(E_{ss}(v_i), v_i)}{\sum_{i\notin\mathscr{F}(E)} \Pi(E_i, v_i)}. \end{aligned} \tag{54}$$

It is straightforward to verify that $\bar{\varphi}$ satisfies the constraint on the minimum total generated power. In fact, for an arbitrary E, it holds:

$$\sum_{i\notin\mathscr{F}(E)} \bar{\varphi}_i(E) \geq \sum_{i\notin\mathscr{F}(E)} \Pi(E_i, v_i) \cdot r(E) = P_L - \sum_{i\in\mathscr{F}(E)} \bar{\varphi}_i(E). \tag{55}$$

We define now $\bar{E}(\cdot)$ as the unique solution, for $i = 1, \dots, N$, of the following system of differential equations:

$$\dot{E}_i(t) = \Pi(E_i(t), v_i) - \varphi_i(E(t)) \qquad E_i(0) = E_i^0. \tag{56}$$

The corresponding power profile will be equal to the feedback law $\bar{\varphi}$ evaluated along $\bar{E}$, with $\bar{P}_i(t) = \bar{\varphi}_i(\bar{E}(t))$ for $i = 1, \dots, N$. To show that $\bar{E}$ satisfies the final state condition in (36), considering that $\dot{\bar{E}}_i(t) = 0$ if $\bar{E}_i(t) = E_{ss}(v_i)$, it is sufficient to verify the following:

$$\dot{\bar{E}}_i(0) > 0 \qquad \forall i \notin \mathscr{F}(E^0) \tag{57}$$

$$\frac{\partial \zeta_i(E)}{\partial E_j} > 0 \qquad \begin{matrix} \forall i, j \notin \mathscr{F}(E) \\ \forall E \in \prod_i \left(E_i^0, E_{ss}(v_i)\right] \end{matrix} \tag{58}$$

where $\zeta_i(E)$ denotes the time derivative of the i-th component of E when the feedback $\bar{\varphi}$ is applied. Specifically, for $i \notin \mathscr{F}(E)$, it holds:

$$\zeta_i(E) = \begin{cases} \Pi(E_i, v_i)\left[1 - r(E)\right] \text{ if } & \Pi(E_i, v_i) \cdot r(E) > P_{MIN} \\ \Pi(E_i, v_i) - P_{MIN} & \text{if } \Pi(E_i, v_i) \cdot r(E) \leq P_{MIN}. \end{cases} \tag{59}$$

For the inequality in (57) notice that $\dot{\bar{E}}_i(0) = \zeta_i(E^0)$. If one replaces E^0 in (59), for the case $\Pi(E_i, v_i) \cdot r(E) > P_{MIN}$ it is sufficient to consider that $r(E^0) < 1$ since $P_L < \Pi_{TOT}(E^0)$. The inequality in the other case is verified from (37). For condition (58), this is always satisfied when $\Pi(E_i, v_i) \cdot r(E) \leq P_{MIN}$ since $\Pi_E > 0$. In the opposite case, it holds $r(E) > 0$ since $\Pi(E_i, v_i)$ and P_{MIN} are both nonnegative quantities. Furthermore, for the considered E, we have $P_L < \Pi_{TOT}(E^0) < \Pi_{TOT}(E)$ and therefore $r(E) < 1$. From expression (59) and the positivity of Π_E, condition (58) holds if the following inequality is satisfied:

$$\frac{\partial r(E)}{\partial E_j} = -\frac{\Pi_E(E_j, v_j)\left[P_L - \sum_{i\in\mathscr{F}(t)} \Pi(E_{ss}(v_i), v_i)\right]}{\left[\Pi_{TOT}(\bar{E}(t)) - \sum_{i\in\mathscr{F}(t)} P_i(t)\right]^2} < 0. \tag{60}$$

This is true since the positivity of $\left[P_L - \sum_{i\in\mathscr{F}(t)} \Pi(E_{ss}(v_i), v_i)\right]$ follows from the fact that $r(E)$ and its denominator in (54) are both greater than zero. The proof is concluded by noticing that also the constraints in (36) on the single power $\bar{P}_i$ are satisfied since $\varphi_i(E) \geq P_{MIN}$ by definition and $r(E) < 1$.

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